Turbulent Flows
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Solution to Exercise 13.10

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In Table 13.2 the transfer function of the sharp spectral filter is given as

$$\hat{G}(\kappa) = \begin{cases} 0 \text{ if } |\kappa| \geq \kappa_c \\ 1 \text{ if } |\kappa| < \kappa_c. \end{cases} \quad (1)$$

Together with the Kolmogorov spectrum (Eq.(6.239)) we have from Eq.(13.65)

$$\langle k_r \rangle = \int_{\kappa_c}^{\infty} C\varepsilon^{2/3}\kappa^{-5/3}\, d\kappa = -\tfrac{3}{2}C\varepsilon^{2/3}\, \kappa^{-2/3}\Big|_{\kappa=\kappa_c}^{\infty} = \tfrac{3}{2}C\varepsilon^{2/3}\kappa_c^{-2/3}, \quad (2)$$

and with the lengthscale $L = k^{3/2}/\varepsilon$ the dissipation rate ε can be eliminated, leading to

$$\frac{\langle k_r \rangle}{k} = \tfrac{3}{2}C(\kappa_c L)^{-2/3}. \quad (3)$$

This is equivalent to Eq.(13.66). If 80% of the spectrum is resolved, 20% are residual motions. Setting Eq.(3) equal to 0.2 and using $C = 1.5$ for the universal Kolmogorov constant (Eq.(6.239) below) leads $\kappa_c L \approx 38$. With $\ell_{EI} = \frac{0.43}{6}L$ for high-Reynolds-number turbulence and $\kappa_c = \pi/\Delta$, the corresponding filter width is

$$\frac{\Delta}{\ell_{EI}} = \frac{6\pi}{0.43\kappa_c L} \approx 1.16. \quad (4)$$

In Table 13.2 the transfer function of the Gaussian filter is given,

$$\hat{G}(\kappa) = \exp\left(-\frac{\kappa^2\Delta^2}{24}\right). \quad (5)$$

Together with the Kolmogorov spectrum one gets from Eq.(13.65)

$$\langle k_r \rangle = \int_0^{\infty} \left[1 - \exp\left(-2\frac{\kappa^2\Delta^2}{24}\right)\right] C\varepsilon^{2/3}\kappa^{-5/3}\, d\kappa$$

$$
\begin{aligned}
&= C\varepsilon^{2/3}\int_0^\infty \left[1-\exp\left(-\frac{\kappa^2\Delta^2}{12}\right)\right]\kappa^{-5/3}\,d\kappa \\
&= C\varepsilon^{2/3}\int_0^\infty (1-e^{-x})\left(\frac{\sqrt{12x}}{\Delta}\right)^{-5/3}\frac{\sqrt{12}}{2\Delta}\frac{dx}{\sqrt{x}} \\
&= kC96^{-1/3}\left(\frac{\Delta}{L}\right)^{2/3}\int_0^\infty (1-e^{-x})x^{-4/3}\,dx. \qquad (6)
\end{aligned}
$$

On the third line of Eq.(6), κ was substituted by $x=\kappa^2\Delta^2/12$ and on the fourth line, ε was eliminated using $L=k^{3/2}/\varepsilon$. Dividing Eq.(6) through k leads Eq.(13.69). In Eq.(6), if Δ is replaced by π/κ_c ($\kappa_c=\pi/\Delta$) and if the integral is substituted by I_0 (Eq.(13.70)) we get

$$
\kappa_c L = \pi\left(\frac{5}{96^{1/3}}CI_0\right)^{3/2}. \qquad (7)
$$

Numerical evaluation leads $I_0 \approx 4.06235$ and together with $C=1.5$, $\kappa_c L \approx 54$ is resulting. The corresponding filter width is then from Eq.(4) $\Delta/\ell_{EI} \approx 0.813$.